Signatures of Stable Multiplicity Spaces in Symmetric Group Restrictions

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PRIMES-USA, May 21, 2016

Permutations and Two Row Notation

Definition

A **permutation** is a bijection from the set $\{1, 2, ..., n\}$ to itself. S_n is the set of permutations of size n.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 6 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 1 & 4 & 2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 5 & 4 \end{pmatrix}$$

Cycle Notation

Definition

A permutation is written in **cycle notation** by writing cycles with each number followed by the number it maps to.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 1 & 4 & 2 & 5 \end{pmatrix} = (1\ 3)(2\ 6\ 5)(4)$$

$$(1\ 2\ 5\ 4\ 6)(3) \circ (1\ 3)(2\ 6\ 5)(4) = (1\ 3\ 2)(4\ 6)(5)$$

Adjacent Transpositions

Definition

The **adjacent transpositions** are the transpositions $(1\ 2), (2\ 3), ..., (n-1\ n)$. Any permutation can be written as a product of adjacent transpositions.

$$(5\ 6)(4\ 5)(3\ 4)(2\ 3) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 3 & 4 & 5 \end{pmatrix}$$

$$(3\ 4)\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 4 & 3 & 5 \end{pmatrix}$$

$$(2\ 3)(1\ 2)\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 4 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 1 & 5 & 2 & 5 \end{pmatrix}$$

Representations

Definition

A **representation** of S_n is a map from permutations to matrices that multiply the same way as the permutations.

Definition

Irreducible representations are representations with no smaller representations inside them. Any representation can be broken down into irreducible.

Permutation Representation of S_n

Definition

The **permutation representation** of S_n is a space of dimension n, where permutations act by permuting the basis vectors.

For $n = 5$, the permutations act on a space of ordered 5-tuples:

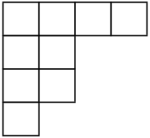
$$(23)(x_1, x_2, x_3, x_4, x_5) = (x_1, x_3, x_2, x_4, x_5)$$

Young Diagrams

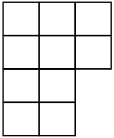
Definition

A **partition** of a positive integer n is a way to write n as an unordered sum of positive integers. A **Young diagram** is a diagram of n squares where each row contains at most as many squares as the row above.

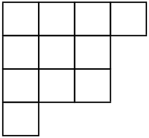
Each Young diagram with n squares corresponds uniquely to a partition of n.



$9 = 4 + 2 + 2 + 1$



$10 = 3 + 3 + 2 + 2$



$11 = 4 + 3 + 3 + 1$

Young Tableaux

Definition

A **Young tableau** is a way to fill in the squares of a Young diagram with the numbers 1 through n, each occurring once. A **standard Young tableau** is a Young tableau where the numbers are increasing in each row and column.

5	9	1	4
7	8		
6	2		
3			

2	5	7	11
3	4	9	
8	6	10	
1			

Generic Young tableaux

1	3	5	9
2	6		
4	7		
8			

1	3	4	7
2	5	8	
6	10	11	
9			

Standard Young tableaux

Standard Column Tableaux

Definition

The **standard column tableau** for a Young diagrams is formed by filling numbers along columns.

1	5	9	12
2	6	10	
3	7	11	
4	8		

1	4	6	8
2	5	7	
3			

Irreducible Representations of S_n

Theorem

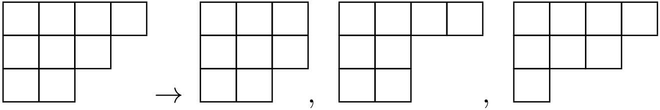
Irreducible representations *of S_n correspond to Young diagrams of size n, or partitions of n.*

Theorem

Standard Young tableaux *form a basis of this representation.* ***Adjacent transpositions*** *act by the identity if the numbers are in the same row, -1 if they are in the same column, and a linear combination of the original and new standard Young tableaux otherwise.*

Theorem

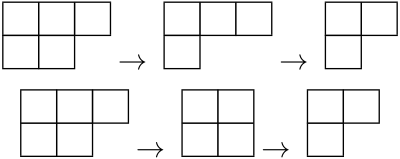
*A **restriction** of an irreducible representation of S_n to S_{n-1} is formed by removing a box from the Young diagram. A restriction from S_n to S_{n-k} is formed by removing k boxes.*



Multiplicity Spaces

Theorem

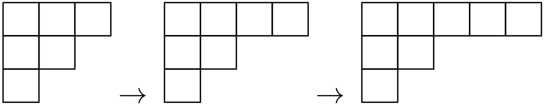
A ***multiplicity space*** *counts the number of copies of a fixed irreducible representation of S_{n-k} inside a fixed irreducible representation of S_n. The dimension equals the number of orders to remove the k boxes from the larger tableau.*



Multiplicity Space Stable Sequence

Definition

A **stable sequence** of representations is formed by starting from a Young tableau and adding boxes to the first row. A stable sequence of multiplicity spaces is formed by doing this for both tableaux.



Inner Products and Norms

Definition

An **inner product** on a vector space is a bilinear function from pairs of vectors to $\mathbb{R}$. Two vectors are **orthogonal**, or perpendicular, if their inner product is 0.

Definition

The **norm** of a vector is the inner product of the vector with itself.

Invariant Inner Product On Multiplicity Spaces

Definition

The **invariant inner product** on an irreducible representation of S_n is chosen such that the column tableau has norm 1 and norms are invariant under S_n.

Definition

The invariant inner product on a **multiplicity space** is defined by **dividing** the invariant inner product on the larger representation by the invariant inner product on the smaller representation.

Multiplicity Space Basis

Definition

Basis vectors of a multiplicity space are standard Young tableaux where numbers are entered in the order boxes are added, starting from the standard column tableau for the smaller representation.

1	4	6
2	5	
3		

$\rightarrow$

1	4	6
2	5	7
3		

$\rightarrow$

1	4	6
2	5	7
3	8	

1	4	6
2	5	
3		

$\rightarrow$

1	4	6
2	5	
3	7	

$\rightarrow$

1	4	6
2	5	8
3	7	

Content and Length

Definition

The **content** a_i of a number i in a Young tableau is the column number of i minus the row number.

1	2	4	6
3	5		

$$a_1 = 1 - 1 = 0,\ a_2 = 2 - 1 = 1,\ a_3 = 1 - 2 = -1$$

Definition

The **length** of a standard Young tableau measures the distance from the column tableau.

Norms Under Adjacent Transpositions

Theorem

Given two tableaux $T' = (i \quad i+1)T$ *such that* T' *has smaller length, the norms for the corresponding basis vectors* v_T *and* $v_{T'}$ *satisfy*

$$\|v_T\|^2 = \frac{(a_{i+1} - a_i)^2}{(a_{i+1} - a_i + 1)(a_{i+1} - a_i - 1)} \cdot \|v_{T'}\|^2.$$

Non-First Row Case Example

1	4	6	7
2	5		
3			

$\rightarrow$

1	3	6	7
2	5		
4			

$\rightarrow$

1	3	6	7
2	4		
5			

$\rightarrow$

1	3	5	7
2	4		
6			

$\rightarrow$

1	3	5	6
2	4		
7			

$\rightarrow$

1	3	4	6
2	5		
7			

$\rightarrow$

1	3	4	5
2	6		
7			

Non-First Row Case Formula

Theorem

If no boxes are added to the first row, all the basis vectors for given Young tableau shapes have the same norm up to a positive constant, namely

$$\prod_{i=0}^{k-1} \frac{n-m+A_m-A_{c_i}}{n-m+A_m-A_{c_i}+1},$$

Where $c_0, c_1, ...c_{k-1}$ *are the entries of the added boxes in the standard column tableau and* A_i *is the content of* i *in the standard column tableau.*

First Row Case Formula

Theorem

Suppose we are adding k *boxes total,* j *of which are in the first row, and the* $k-j$ *others have entries* $c_1, c_2, ..., c_{k-j}$ *in the standard column tableau. The basis vector with entries* $n-d_1 > n-d_2 > ... > n-d_{k-j}$ *in the boxes occupied by* $c_1, c_2, ..., c_{k-j}$*, respectively, in the standard column tableau has norm*

$$\prod_{i=1}^{k-j} \frac{n-d_i+i-1-m+A_m-A_{c_i}}{n-d_i+i-m+A_m-A_{c_i}}.$$

Signatures and Generalizations

Definition

The **signature** for an inner product on a vector space is the number of basis vectors with positive norm minus the number with negative norm. The signature is independent of the choice of orthogonal basis.

Because the norms are polynomials, we can extend the computation of norms and signatures to arbitrary complex values even if the generalized group S_t for $t \in \mathbb{C}$ has no concrete meaning.

Acknowledgements

I sincerely thank my mentor, Siddharth Venkatesh, for his guidance and advice regarding this project. I thank Dr. Tanya Khovanova for her help preparing this talk. I thank Professor Pavel Etingof for the problem suggestion. I thank the PRIMES-USA program for giving me the opportunity to experience math research.